A COMBINATORIAL PROPERTY OF ABELIAN SEMIGROUPS

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A well-known result due to van der Waerden asserts that given a finite partition of $\mathbb{N}$, one of the subsets contains arbitrarily long finite arithmetic progressions. We shall show that actually all abelian semigroups play a similar combinatorial property. Our approach makes use of techniques from topological dynamics.

In what follows $\mathcal{S}$ will denote an additive semigroup.

By a *configuration* in $\mathcal{S}$ we shall mean any ordered finite subset of $\mathcal{S}$. We shall say that a configuration $Q = \{y_1, \ldots, y_p)$ *mimics* (at the left) the configuration $P = \{x_1, \ldots, x_p)$ provided there exist z in $\mathcal{S}$ and n in $\mathbb{N}^*$ such that

$$y_k = z + n x_k, \text{ for all } k.$$

If P is an arithmetic progression and Q mimics P, then Q is also an arithmetic progression.

We shall say that the semigroup $\mathcal{S}$ plays *the van der Waerden property* (abbreviated (W)) provided the following condition is fulfilled:

(W) *For every configuration P in $\mathcal{S}$ and every finite partition*

$$\mathcal{S} = C_1 \cup \ldots \cup C_r$$

of $\mathcal{S}$, one of the sets C_k contains a configuration that mimics P.

Then van der Waerden result is simply $\mathbb{N}$ plays (W). The extension of this fact to $\mathbb{N}^k$ was done by Gallai [R] and Witt [W].

For $\mathcal{S} = \mathbb{T}$, the property (W) means that for each family $\{z_1, \ldots, z_p\}$ of elements of $\mathbb{T}$ there exist a point z in $\mathbb{T}$ and a sequence $(k_n)_n$ of naturals such that $k_n \to \infty$ and

$$z_1^{k_n} \to z, \ \ldots, \ z_p^{k_n} \to z.$$

Notice that in the presence of torsion, the property (W) might be trivial. In fact, all periodic groups (in particular, all finite groups) play (W). Since (W) is closed under directed unions, we can easily find examples of nonperiodic noncommutative groups playing (W).

Amenable groups can be characterized by a combinatorial property (the so-called *Fölner condition*) which in the discrete case reads as follows:

(FC) *For every* $\varepsilon > 0$ *and every finite subset* K *of the group* G *there exists a nonvoid finite subset* U *of* G *such that* card $(xU \Delta U) < \varepsilon$ card U, *for every* x *in* K.

Amenability depends on the topology we consider on the given group, e.g. discrete SO(3) is not amenable while SO(3) with respect to the natural topology is compact and thus amenable. See [G]. Therefore, amenability and (W) are distinct properties.

Our main result about (W) is as follows:

THEOREM. *Every abelian semigroup* $\mathcal{S}$ *plays the van der Waerden property.*

The proof depends on the Birkhoff recurrence theorem, extended by Furstenberg for families of commuting actions:

LEMMA (see [F]). *Let* X *be a compact metric space and let* $T_1, \ldots, T_p$ *be pairwise commuting homomorphisms of* X. *Then there exist sequences* $(k_n)_n$ *of naturals going to* ∞ *and points* x *in* X *such that*

$$T_1^{k_n} x \to x, \ \ldots, \ T_p^{k_n} x \to x.$$

Proof of Theorem. Let $P = \{x_1, \ldots, x_p\}$ be a configuration in $\mathcal{S}$ and let $\mathcal{S} = C_1 \cup \ldots \cup C_r$ be a finite partition of $\mathcal{S}$. Consider the compact Hausdorff space $\Omega = \{1, \ldots, r\}^{\mathcal{S}}$ (endowed with the product topology) and the point ξ of Ω given by

$$\xi(x) = k \text{ iff } x \in C_k.$$

The mappings $T_k : \Omega \to \Omega$ ($k \in \{1, \ldots, p\}$) given by

$$(T_k\omega)(x) = \omega(x + x_k)$$

constitute pairwise commuting homomorphisms on Ω.

The smallest closed subset X of Ω, containing ξ and invariant under $T_1, \ldots, T_p$ is precisely the closure of the sequence of translates of ξ, i.e.

$$X = \overline{\left\{\xi\left(x + \sum n_k x_k\right) \middle| \, n_k \in \mathbb{N}, \, k \in \{1, \ldots, p\}\right\}}.$$

Consequently, X is compact and separable and thus it is metrisable . By the Lemma above (applied to the mappings $T_k \mid X$) one can obtain a point $\eta \in X$ and a number $n \in \mathbb{N}^*$ such that

$$T_1^n \eta(0) = \ldots = T_p^n \eta(0)$$

i.e. $\eta(nx_1) = \ldots = \eta(nx_p)$. Taking into account the definition of X, one finds a z in $\mathcal{S}$ such that

$$\xi(z + nx_1) = \ldots = \xi(z + nx_p).$$

Letting k the common value of ξ, the later means that $z + nx_1, \ldots, z + nx_p$ belong to C_k. ■

By using the above Theorem one can prove results of the following type:

COROLLARY. *Suppose* $A_1, \ldots, A_p$ *are* $m \times n$*-dimensional matrices with entries in a field* K. *Then there exist a matrix* $A \in M_{m,n}(K)$ *and a number* $q \in \mathbb{N}^*$ *such that* $A + qA_1, \ldots, A + qA_p$ *have the same rank and the same index.*

The free semigroup with two generators a and b is not abelian and does not play (W). Think of the configuration $\{a,b\}$ and the partition $C_1 \cup C_2 \cup C_3 \cup C_4$, where

C_1 consists of $\varnothing$ and the words of the form $a \ldots a$

C_2 consists of the words of the form $a \ldots b$

C_3 consists of the words of the form $b \ldots a$

C_4 consists of the words of the form $b \ldots b$.

We do not know whether (W) passes to subsemigroups; it is clear that it passes to quotients. In this connection we state the following:

Problem. *Suppose* $\mathcal{S}$ *is a torsion free semigroup with the van der Waerden property. Is* $\mathcal{S}$ *necessarily abelian? If not, what reasonable condition should be added to get commutativity?*

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